Roll No. ....

DD–2801

M. A./M. Sc. (Previous) EXAMINATION, 2020

MATHEMATICS

Paper First

(Advanced Abstract Algebra)

Time : Three Hours

Maximum Marks : 100

Note : Attempt any *two* parts from each question. All questions carry equal marks.

Unit—I

1. (a) Prove that any finite P-group is solvable.

   (b) E be an extension field of a field F and $u \in E$ be algebraic over F. If $p(x) \in F(x)$ be a polynomial of the least degree such that $p(u) = 0$, then prove that :

   (i) $p(x)$ is irreducible over F

   (ii) If :

   $$g(x) \in F(x)$$

   is such that $g(u) = 0$, then $p(x) \mid g(x)$.

(iii) There is exactly are menic polynomial $p(x) \in F(x)$ is least degree such that $p(u) = 0$.

(c) If C is the field of complex numbers and R is the field of real numbers, then show that C is a normal extension of R.

Unit—II

2. (a) Show that the polynomial :

$$x^7 - 10x^5 + 15x + 5$$

is not solvable by radicals over Q.

(b) Prove that :

$$f(x) \in F(x)$$

is solvable by radicals over F if and only if its splitting field E over F has solvable Galois group G (E/F).

(c) Let F be a field of characteristic $\neq 2$ and $x^2 - a \in F(x)$ be an irreducible polynomial over F, then prove that its Galois group is of order 2.

Unit—III

3. (a) Let R be a ring with unity. Show that an R module M is cyclic if and only if $M \cong \frac{R}{I}$ for some left ideal I of R.

(b) If M be a finitely generated free module over a commutative ring R. Then prove all bases of M have the same number of elements.

(c) Prove that every submodule and every quotient module of Noetherian module is Noetherian.

Unit—IV

4. (a) Let $\lambda \in F$ be a characteristic root of $T \in A(V)$. Then prove that for any polynomial $q(x) \in F(x)$, $q(\lambda)$ is a characteristic root of $q(T)$.

(b) Let the linear transformation $T \in A_F(V)$ be nilpotent, then prove that :

$$\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \ldots\ldots\ldots\ldots + \alpha_m T^m$$

where $\alpha_i \in F, 0 \le i \le m$, is invertible if $\alpha_0 \ne 0$.

(c) Find the Jordan Canonical form of :

$$A = \begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}.$$

Unit—V

5. (a) Find the rational canonical form of the matrix whose invariant factors are $(x-3),(x-3)(x-1)$ and $(x-3)\,(x-1)^2$.

(b) Find the Smith normal form and rank for the matrix over PID R :

$$\begin{bmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{bmatrix},$$

where $R = Q\,[x]$.

(c) Let R be a principal ideal domain and let M be any finitely generated R module, then :

$$M \simeq R^s \oplus \frac{R}{R\,a_1} \oplus \frac{R}{R\,a_2} \oplus \dots\dots \oplus \frac{R}{R\,a_r}$$

a direct sum of cyclic modules where the a_i are non-zero non-units and $a_i \mid a_{i+1}, i = 1, 2, \dots\dots, r-1$.